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CHARACTERIZATIONS OF A HIRATA SEPARABLE GROUP RING

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Abstract

Let R be a ring with 1, G be a group, RG be an Azumaya group ring of G over R, K be a subgroup of G, and D be the double centralizer of RK in RG. Characterizations of a Hirata separable extension RG over D, and D is a direct summand as a D-bimodule are given.

1. Introduction

Let R be a ring with 1, G be a group, and RG be a group ring of G over R. DeMeyer and Janusz ([2], Theorem 1) showed that RG is an Azumaya algebra, if and only if (1) R is an Azumaya algebra, (2) the center of G has a finite index, and (3) the order of the commutator subgroup of G is a finite integer and invertible in R. When R is a field and G is a finite group of a nonzero order in R, Hirata ([3], Proposition 6) proved that for any subgroup K of G, RG is a Hirata separable extension of the double centralizer of RK in RG. This fact was extended to any Azumaya group ring RG ([8], Lemma 4.2). Noting that in the above Hirata theorem, the

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double centralizer D of RK in RG is a direct summand of RG as a D-bimodule ([8], Lemma 4.2), in the present paper, we shall characterize a subgroup K such that RG is a Hirata separable extension of the double centralizer D of RK in RG, and D is a direct summand of RG as a D-bimodule. Also, some equivalent conditions are given for a subgroup K of G such that RKC is a separable subalgebra of RG.

2. Basic Definitions and Notations

Let B be a ring with 1 and A be a subring of B with the same identity 1. Then B is called a *separable extension* of A, if there exist $\{a_i, b_i$ in B, $i = 1, 2, \ldots, k$ for some integer $k\}$ such that $\sum a_i b_i = 1$ and $\sum x a_i \otimes b_i = \sum a_i \otimes b_i x$ for all x in B, where $\otimes$ is over A. In particular, B is called an *Azumaya algebra*, if it is a separable extension over its center. A ring B is called a *Hirata separable extension* of A, if $B \otimes_A B$ is isomorphic to a direct summand of a finite direct sum of B as a B-bimodule. For more about Azumaya algebras and Hirata separable extensions, see [6], [7], and [9]. Let S be a subset of B. We denote B^S, the centralizer of S in B.

Throughout this paper, R will be a ring with identity 1, R_0 be the center of R, G be a group, RG be a group ring of G over R, and C be the center of RG.

3. Hirata Separable Extensions

Let RG be an Azumaya group ring. We shall characterize a subgroup K of G, such that RG is a Hirata separable extension of the double centralizer D of RK in RG, and D is a direct summand of RG as a D-bimodule. We begin with the Hirata theorem for an Azumaya group algebra over a field and its extension to an Azumaya group ring.

Theorem 3.1 ([3], Proposition 6)**.** *Assume G is a finite group and k is a field of characteristic not dividing the order of G. Then for any subgroup K of G, RG is a Hirata separable extension of the double centralizer of RK.*

Theorem 3.2 ([8], Lemma 4.2)**.** *Let RG be an Azumaya group ring. If K is a finite subgroup of G of order $|K|$ such that $|K|^{-1} \in R$, then* (i) *$(RC)K$ and $(R_0G)^K$ are direct summands of RG as bimodules over themselves*, *and* (ii) *RG is a Hirata separable extension of $(RC)K$ and $(R_0G)^K$, respectively.*

The above Hirata theorem can be strengthened by showing that the Hirata separable extension of the double centralizer of RK as given in Theorems 3.1 and 3.2 is also a progenerator left module over the double centralizer of RK.

Theorem 3.3. *By keeping the notations as given in Theorem* 3.2, *RG is a progenerator module over the double centralizer of RK.*

Proof. Since $|K|^{-1} \in R$, RK is a separable extension of R. By hypothesis, RG is an Azumaya algebra over its center C, so R is an Azumaya algebra over its center R_0 ([2], Theorem 1). Hence, RK is a separable R_0-algebra by the transitivity property of separable extensions. Thus, $C \otimes_{R_0} RK$ is a separable C-algebra; and so as a homomorphic image of $C \otimes_{R_0} RK$, RCK is a separable C-algebra. Therefore, RG is projective over RCK by the lifting property of a projective module over a separable algebra ([1], Proposition 2.3, page 48). Also, since RG is an Azumaya C-algebra and $C \subset RCK$, RG is finitely generated over RCK. Moreover, by Theorem 3.2, RCK is a direct summand of RG as a RCK-bimodule. Hence, RG is a generator over RCK. Thus, RG is a progenerator over RCK. Next, we show that RCK is the double centralizer of RK. In fact, since RCK is a separable subalgebra of the Azumaya algebra RG, $V_{RG}(V_{RG}(RCK)) = RCK$ by the double centralizer property for Azumaya algebras ([1], Theorem 4.3, page 57). But, $V_{RG}(RK) = V_{RG}(RCK)$, so RCK is the double centralizer of RK. This completes the proof.

Next, we characterize the subgroup K such that RG is a Hirata separable extension of the double centralizer D of RK in RG, and D is a direct summand of RG as a D-bimodule. The centralizer of a subset S in RG is denoted by $(RG)^S$.

Theorem 3.4. *Let RG be an Azumaya group ring, C be the center of RG, R_0 be the center of R, K be a subgroup of G, and D be the double centralizer of RK in RG. Then the following are equivalent*:

(1) *RG is a Hirata separable extension of the double centralizer D and D is a direct summand of RG as a D-bimodule*;

(2) *$(R_0G)^K$ is a separable subalgebra of RG*;

(3) *RG is a projective left $(R_0G)^K$-module and $(R_0G)^K$ is a direct summand of RG as a $(R_0G)^K$-bimodule*;

(4) *RG is a Hirata separable extension of $(R_0G)^K$ and $(R_0G)^K$ is a direct summand of RG as a $(R_0G)^K$-bimodule*;

(5) *D is a separable subalgebra of RG*;

(6) *RG is a projective left D-module and D is a direct summand of RG as a D-bimodule.*

Proof. (1) $\Rightarrow$ (2). Since $(RG)^{RK} = (R_0G)^K$ and D is the double centralizer of RK in RG, the centralizer of D is $(R_0G)^K$. Hence, $(R_0G)^K$ is a separable subalgebra of RG ([5], Proposition 1.3).

(2) $\Rightarrow$ (3). Since RG is an Azumaya algebra and $(R_0G)^K$ is a separable subalgebra of RG, $(R_0G)^K$ is a direct summand of RG as a $(R_0G)^K$-bimodule. Also, the projective module RG over C is lifted to the left $(R_0G)^K$-module by the lifting property of a projective module over a separable algebra ([1], Proposition 2.3, page 48), so RG is a left projective module over $(R_0G)^K$.

(3) $\Rightarrow$ (4). Since RG is an Azumaya algebra and a projective left $(R_0G)^K$-module, RG is a Hirata separable extension of $(R_0G)^K$ ([4], Theorem 1).

(4) $\Rightarrow$ (5). Since RG is a Hirata separable extension of $(R_0G)^K$ and $(R_0G)^K$ is a direct summand of RG as a $(R_0G)^K$-bimodule, the centralizer of $(R_0G)^K$ is a separable subalgebra of RG ([5], Proposition 1.3). Noting that D is the centralizer of $(R_0G)^K$, we conclude that D is a separable subalgebra of RG.

(5) $\Rightarrow$ (6). Since RG is an Azumaya algebra and D is a separable subalgebra of RG, the statement (6) holds by the argument of (2) $\Rightarrow$ (3).

(6) $\Rightarrow$ (1). Since RG is an Azumaya algebra and a projective left D-module, RG is a Hirata separable extension of D ([4], Theorem 1).

For a subgroup K with an invertible order in R, RK is a separable extension of R. But, R is an Azumaya R_0-algebra, so RK is a separable R_0-algebra. Hence, RCK is a separable subalgebra of RG. Thus, RCK is the double centralizer of RK in RG by the double centralizer theorem for Azumaya algebras. Therefore, K satisfies Theorem 3.4; and so the following corollary holds.

Corollary 3.5. *Let RG be an Azumaya algebra, C be the center of RG, Z be the center of G, and K be a subgroup of G with an invertible order. Then* (1) *RCK is the double centralizer of RK and $R(ZK)$, respectively, and* (2) *RG is a projective Hirata separable extension of RCK.*

Corollary 3.6. *Let $\alpha_0 : K \to (R_0G)^K$, where R_0 is the center of R and K is a subgroup of G. If RG is an Azumaya algebra, where G is an infinite non-abelian group, then α_0 is not one-to-one.*

Proof. Since RG is Azumaya, the commutator subgroup G' of G is finite, and the center Z of G is infinite ([2], Theorem 1). Hence, $G' \neq G'Z$. But $\alpha_0(G') = (R_0G)^{G'} = (R_0G)^{G'Z} = \alpha_0(G'Z)$, so α_0 is not one-to-one.

In Theorem 3.4, the class of subgroups K of G is characterized so that RG is a Hirata separable extension of the double centralizer D of RK in RG, and D is a direct summand of RG as a D-bimodule. This class contains subgroups K with an invertible order by Theorems 3.2 and 3.3.

Let Z be the center of G. Then KZ is also in the class, whenever K is in the class because $(RG)^K = (RG)^{KZ}$ by Theorem 3.4. Hence, the class contains subgroups KZ of order not necessarily finite invertible. Next, we give some equivalent conditions for a subgroup K of G such that RKC is a separable C-subalgebra of RG. We shall employ a property of a subgroup K of G as given in [8].

Lemma 3.7 ([8], Lemma 3.2)**.** *Let K be a subgroup of G. If RG is Azumaya, then RK is Azumaya.*

Theorem 3.8. *Let RG be an Azumaya algebra, C be the center of RG, K be a subgroup of G, and C_K be the center of RK. Then the following are equivalent*:

(1) *RKC is a separable subalgebra of RG*;

(2) *C_KC is a separable subalgebra of RG*; *and*

(3) *RG is a projective left C_KC-module and C_KC is a direct summand of RG as a C_KC-bimodule.*

Proof. (1) $\Leftrightarrow$ (2). By Lemma 3.7, RK is an Azumaya C_K-algebra, so $RK \otimes_{C_K} C_KC$ is an Azumaya C_KC-algebra. Hence, $RK \otimes_{C_K} C_KC \cong RKC$ as Azumaya C_KC-algebras by the multiplication homomorphism; and so RKC is a separable C-subalgebra of RG, if and only if C_KC is a separable C-algebra ([1], Theorem 3.8, page 55). Thus (1) $\Leftrightarrow$ (2).

(2) $\Rightarrow$ (3). Since RG is an Azumaya C-algebra, RG is a projective C-module. But, C_KC is a separable C-algebra by hypothesis, so RG is a projective C_KC-module by the lifting property of projective module over a separable algebra ([1], Proposition 2.3, page 48). Also, C_KC is a separable subalgebra of the Azumaya algebra RG, so C_KC is a direct summand of RG as a C_KC-bimodule.

(3) $\Rightarrow$ (2). By the argument as given in the proof of Theorem 3.8 on page 55 in [1].

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